Deformed Heisenberg algebra with upper bound of momentum value

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Abstract

We consider one dimensional deformed Heisenberg algebra leading to existence of minimal length for coordinate operator and minimal and maximal uncertainty of momentum operator. For this algebra an exactly solvable Hamiltonian is constructed.

1 Introduction

Several independent lines for investigation of matter properties at high energies (string theory [1], black hole physics [2], etc., also see [3]) propose that uncertainty of coordinate ΔX depends on uncertainty of momentum ΔP in such a way

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right). \tag{1}$$

Minimizing the right part of this expression one obtains that uncertainty of coordinate is always larger than some threshold value $\Delta X_{min} = \hbar\sqrt{\beta}$.

Kempf showed that expression (1) could be derived using Heisenberg uncertainty inequality from deformed commutation relation [4]

$$[X, P] = i\hbar(1 + \beta P^2). \tag{2}$$

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It was shown that there existed states for which $\Delta X < \Delta X_{min}$, but they are formal states. That is, mean value of kinetic energy $\langle P^2/2m \rangle$ does not exist for these states [5]. Note, that there exist other algebras approximately leading to the inequality (1) [6, 7], but Kempf's algebra is exact, the simplest and the best studied one.

It is well known that for arbitrary hermitian operators A and B the Heisenberg inequality holds

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{2} \langle C \rangle^2 , \tag{3}$$

where $C = [A, B]/i$ is the hermitian operator too. This inequality holds for all states for which mean values of A^2, B^2, AB and BA are defined (more rigid conditions of holding of the inequality (3) may be found in many mathematical papers, see for instance [8]). Everywhere throughout the paper ΔA denotes $\sqrt{\langle (A - \langle A \rangle)^2 \rangle}$.

If someone tries to analyze deformed commutation relation

$$[X, P] = i f(X, P) \tag{4}$$

then many interesting properties from corresponding Heisenberg inequality may arise (e.g., one can discover that a minimal length must be present, as for the case of Kempf algebra (2)). In order to investigate such properties one should consider, for which states the Heisenberg inequality breaks.

We can outline two approaches to such an analysis. The first one is common and it says that breaking states must be considered carefully. They may be physically accepted states if mean values of operators having clear physical sense converge [9]. For an example in [10] states breaking corresponding Heisenberg inequality were considered meaningful since $\langle P^2 \rangle$ converged (it was implicitly assumed that $\langle X \rangle$, $\langle P \rangle$, etc converged too). As it was noted above, for algebra (2) breaking states are considered meaningless since mean value of kinetic energy diverges [5].

The second approach says[1] that commutation relation (4) has physical meaning itself, so Heisenberg inequality must holds for any physically accepted states. All states breaking the inequality are considered to be formal and thus having no physical meaning. In other words, we restrict ourselves to states satisfying corresponding Heisenberg inequality.

[1]This approach was privately pointed to me by Prof. V. M. Tkachuk

Each algebra (4) is characterized by several parameters describing deformation of canonical commutation relation. It is reasonable to assume that *if these parameters tend to zero then eigenvalues of some system tend to corresponding eigenvalues of the same system with undeformed commutation relation*. For algebra (2) all known exactly solvable systems [5, 11, 12, 13] have this property. This just formulated correspondence principle can be used for results verification. We will use it in the third section to decide which eigenvalues belong to spectrum.

This paper is organized as follows. We describe a deformed commutation relation leading to upper bound of momentum uncertainty in the second section. In the third section we write an exactly solvable Hamiltonian down. And end the paper by several concluding remarks.

2 Algebra leading to upper bound of momentum uncertainty

In order to simplify notation and calculus we use $\hbar = 1$, $m = \frac{1}{2}$ further in the paper.

In [7] a large set of different deformed algebras leading to minimal length was presented. They have the following structure

$$[f(X), P] = i(f'(X) + \beta P^2). \tag{5}$$

Here case of $\beta = 0$ corresponds to case of the usual Heisenberg algebra, where $[X, P] = i$. It was shown that there existed minimal length for a wide class of such algebras. For details see [7].

Particular form of such a deformed algebra used function $f(X) = \tanh X$. In this paper we want to present a further generalization of this algebra presented in [7]:

$$[\tanh \alpha X, P] = i\left(\frac{\alpha}{\cosh^2 \alpha X} + \beta P^2 + \delta\right), \quad \beta > 0,\ \delta > 0. \tag{6}$$

An additional term δ appears in our generalization. If someone puts $\beta = 0$, $\delta = 0$ then an canonical commutation relation is recovered. For $\alpha X \ll 1$ one obtains an algebra similar to algebra (2).

Let us make two assumptions that operator $\tanh \alpha X$ is a bounded one and that $\Delta \tanh \alpha X \leq 1$ holds for all states (it would be obvious if X has a complete set of eigenfunctions). Then applying inequality

(3) to commutation relation (6) one obtains the following chain of inequalities

$$\Delta P \geq \Delta \tanh \alpha X \Delta P \geq \frac{1}{2}\left(\Delta \frac{\alpha}{\cosh^2 \alpha X} + \beta \Delta P^2 + \delta\right) \geq \frac{1}{2}\left(\beta \Delta P^2 + \delta\right). \tag{7}$$

From this chain the following constraints on momentum uncertainty can be derived

$$\frac{\delta}{2} \leq \Delta P \leq \frac{2}{\beta}. \tag{8}$$

In a similar way one can deduce that

$$\frac{\delta^2}{4} \leq \left\langle P^2 \right\rangle \leq \frac{4}{\beta^2}. \tag{9}$$

Minimizing inequality (7) the following constraints on coordinate uncertainty appears

$$\sqrt{\beta\delta} \leq \Delta \tanh \alpha X \leq \alpha \Delta X. \tag{10}$$

Existence of lower bounds for uncertainties of momentum and coordinate operators means that eigenstates of these operators satisfying inequality (7) do not exist ($\Delta A = 0$ for eigenstates of operator A).

The inequality (7) breaks if at least one integral among $\left\langle \tanh^2 \alpha X \right\rangle$, $\langle P^2 \rangle$, $\langle P \tanh \alpha X \rangle$ or $\langle \tanh \alpha X P \rangle$ diverges. Operator $\tanh^2 \alpha X$ is a bounded operator, so the first integral must converge for normalizable states. Divergence of $\langle \psi \,| P^2 |\, \psi \rangle$ means that kinetic energy is not well defined in state ψ. We could not analyze the last two integrals separately, but their difference $\langle [\tanh \alpha X, P] \rangle$ contains terms proportional to $\tanh^2 \alpha X$, 1 and P^2. The first and the second operators are bounded ones, the third is kinetic energy operator again. This fact gives a strong evidence that states breaking inequality (7) are nonphysical.

Note, that the same situation occurs in the case of deformed commutation relation (2): mean value of kinetic energy diverges for states for which inequality (1) breaks [5]. Nature of the breaking was established since explicit representation of P and X operators exists for the algebra (2). Contrarily, we do not know explicit representation of X and P operators of the algebra (6). Only approximate representation can be found in the fashion of paper [7] method:

$$X = x, \quad P \approx p + \beta\left(\frac{1}{6\alpha}\{\cosh^2 \alpha x, 4\alpha^2 p + p^3\} - p\right) + \frac{\delta}{2\alpha}\{\cosh^2 \alpha x, p\}, \tag{11}$$

where small operators x and p satisfy conventional commutation relation $[x,p]=i$. In representation (11) operators X and P satisfy relation (6) in linear approximation over parameters β, δ. Such an approximation is valid far small values of x and p (when the first term of approximation (11) is much larger than the second one).

3 Exactly solvable model

It is possible to construct exactly solvable Hamiltonian in the frame of the deformed algebra (6). We use shape-invariance method to build such a model [14]. Let us introduce annihilation-creation operators

$$A_n = i\xi_n P + \eta_n \tanh X, \tag{12}$$

$$A_n^+ = -i\xi_n P + \eta_n \tanh X. \tag{13}$$

Note that we also fix $\alpha = 1$ in algebra (6).

On the basis of these operators we build partner Hamiltonians

$$H_n^- = A_n^+ A_n = (\xi_n^2 - \xi_n\eta_n\beta)P^2 - \frac{\eta_n^2 + \xi_n\eta_n}{\cosh^2 X} + \eta_n^2 - \xi_n\eta_n\delta, \tag{14}$$

$$H_n^+ = A_n A_n^+ = (\xi_n^2 + \xi_n\eta_n\beta)P^2 - \frac{\eta_n^2 - \xi_n\eta_n}{\cosh^2 X} + \eta_n^2 + \xi_n\eta_n\delta. \tag{15}$$

These Hamiltonians have similar form to the so-called Pöschl-Teller Hamiltonian $H = p^2 - 1/\cosh^2 x$ which is an exactly solvable one [14] in undeformed case.

To find spectrum of Hamiltonian H_0^- we construct a chain of such partners:

$$H_{n-1}^+ = H_n^- + \epsilon_n, \tag{16}$$

where ϵ_n is a constant. The following connections between parameters ξ, η and ϵ can be deduced from equation (16)

$$\xi_n = \frac{\xi_{n-1} + \beta\eta_{n-1}}{\sqrt{1+\beta}}, \; \eta_n = \frac{\eta_{n-1} - \xi_{n-1}}{\sqrt{1+\beta}}, \; \epsilon_n = (\eta_{n-1}^2 - \eta_n^2)(1+\delta). \tag{17}$$

Eigenvalues of Hamiltonian H_0^- read

$$E_n = \sum_{i=1}^{n} \epsilon_i = (\eta_0^2 - \eta_n^2)(1+\delta). \tag{18}$$

η_n can be expressed with the help of parameters η_0, ξ_0 of initial Hamiltonian H_0^-

$$\eta_n = \eta_0 \cos n\theta - \xi_0 \frac{1}{\sqrt{\beta}} \sin n\theta, \tag{19}$$

where $\cos\theta = \frac{1}{\sqrt{1+\beta}}$ and $\sin\theta = \sqrt{\frac{\beta}{1+\beta}}$.

Integer quantum number n varies in the range from 0 to n_{max}, where n_{max} is the greatest integer less than $\frac{1}{\theta}\arctan\frac{\eta_0\sqrt{\beta}}{\xi_0}$. This condition was derived from the requirement that

$$\eta_n > 0. \tag{20}$$

If this requirement breaks then the signs in the expression for annihilation operator (12) changes. Here we appeal to the correspondence principle formulated in the Introduction: if η_n changes sign then n^{th} eigenfunction becomes unnormalizable in the undeformed space. In deformed case for $n > n_{max}$ eigenvalues decreases if n increases: it is unusual feature of quantum systems.

The spectrum of Hamiltonian

$$H = P^2 - \frac{30}{\cosh^2 X} \tag{21}$$

is showed on Fig. 1. All levels of deformed problem are above corresponding levels of undeformed one.

In linear approximation over β, δ expression (18) can be simplified to

$$E_n \approx (1+\delta)(\eta_0^2 - (\eta_0 - \xi_0 n)^2) - \frac{\beta}{3}(\eta_0 - \xi_0 n)\left(\xi_0 n^3 - 3\eta_0 n^2 + 2\xi_0 n\right). \tag{22}$$

Hamiltonian in linear approximation is

$$H_0^- \approx \xi_0^2 p^2 - \frac{\eta_0^2 + \xi_0\eta_0}{\cosh^2 x} + \eta_0^2 + \delta\left(\xi_0^2\left\{p, \frac{1}{2}\{p, \cosh^2 x\}\right\} - \xi_0\eta_0\right) +$$
$$+ \beta\left(\xi_0^2\left\{p, \frac{1}{6}\{\cosh^2 x, 4p + p^3\}\right\} - (2\xi_0^2 + \xi_0\eta_0)p^2\right) = H^{lin}. \tag{23}$$

As one should expect $\left\langle \psi_n \middle| H^{lin} \middle| \psi_n \right\rangle$ exactly coincides with expressions (22) for all n, except for $n = n_{max}$. Here ψ_n denotes n^{th} eigenfunction of undeformed Pöschl-Teller system. This exception arises since corresponding integral diverges for large x, where approximation (11) becomes invalid. But the question on the range of n variation needs more attention and it seems it can be resolved finally only if exact representation of algebra (6) is found.

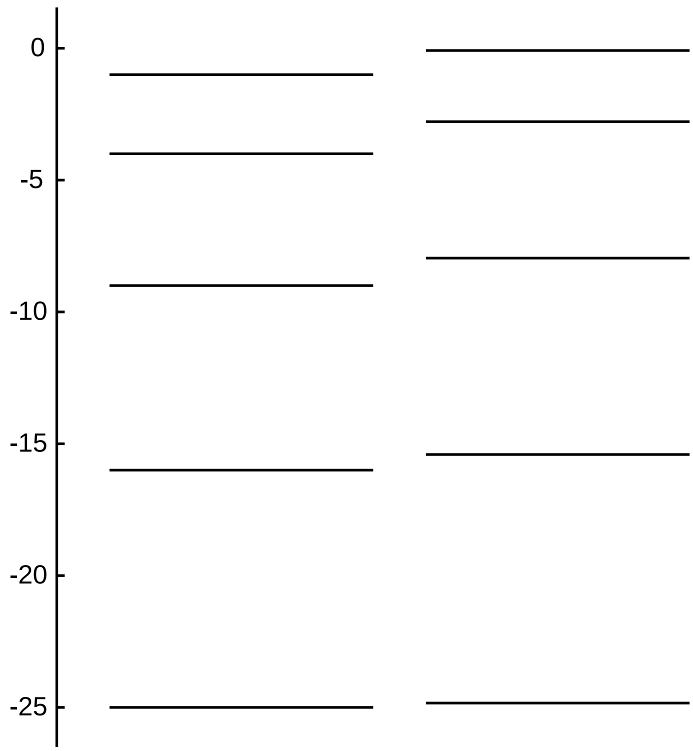

Figure 1: Eigenvalues of Hamiltonian (21) in undeformed case [14] (left column) and in deformed case: equation (18); $\beta = 0.01$, $\delta = 0.01$ (right column).

4 Concluding remarks

Algebra (6) is not a unique algebra characterizing by upper bound for mean values of P^2 operator. There are many other such algebras. One of them reads

$$[X, P] = i(1 + \alpha X^4 + \beta P^2). \tag{24}$$

Using Heisenberg inequality (3) it is easy to show that

$$\frac{16}{\alpha\beta^3} \geq \left\langle P^2 \right\rangle \geq (\Delta P)^2 \geq \frac{4}{9}(3\alpha)^{\frac{1}{2}}, \tag{25}$$

$$\frac{4}{\alpha\beta} \geq \left\langle X^2 \right\rangle \geq (\Delta X)^2 \geq \beta. \tag{26}$$

We have not found explicit representation of X and P operators for it as for algebra (6). We choose algebra (6) for our analysis since exactly

solvable model exists for it and right side of (6) contains bounded operators and P^2 operator having a clear physical sense of kinetic energy.

Existence of X and P representation is still an open question. On one hand, in classical mechanics we always can change variables (X, P) into (x, p) that Poisson bracket $\{x, p\} = 1$ (Darboux theorem, [15]), for which obvious representation exists. On the other hand, if $\beta\delta > 4$ then algebra (6) is self-contradictory (as it follows from constraints (8)). It follows from (26) that if $\alpha\beta^2 > 4$ then algebra (24) is self-contradictory too. Saying that the algebra is self-contradictory we imply that for such values of deformation parameters there are no states satisfying corresponding Heisenberg inequality.

There are some other open questions which are needed to be answered. They are: generalization of such complicated algebras as (6) and (24) to multidimensional case; construction of analogue of Galilei transformation. It is the third open question. Obviously one can answer them (partly or completely) if one finds an explicit representation.

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